All Tangled Up: Geometry of Tangloid Curves

Kyle Kelley and Prof. Carol S. Schumacher

Department of Mathematics and Statistics, Kenyon College, Gambier, Ohio, 43022

Introduction

A **hypotrochoid** is a mathematical curve created by rolling a wheel around the inside of a larger circular track and tracing out the trajectory of a fixed point relative to the center of the wheel (*Spirograph* curves). **Epitrochoids** follow the same convention with the wheel rolling along the outside of the track. We need three parameters to create these curves:

- R is the radius of the circular track.
- r is the radius of the wheel.
- pr is the distance of the pen from the center of the wheel, so p is a percentage of the radius.

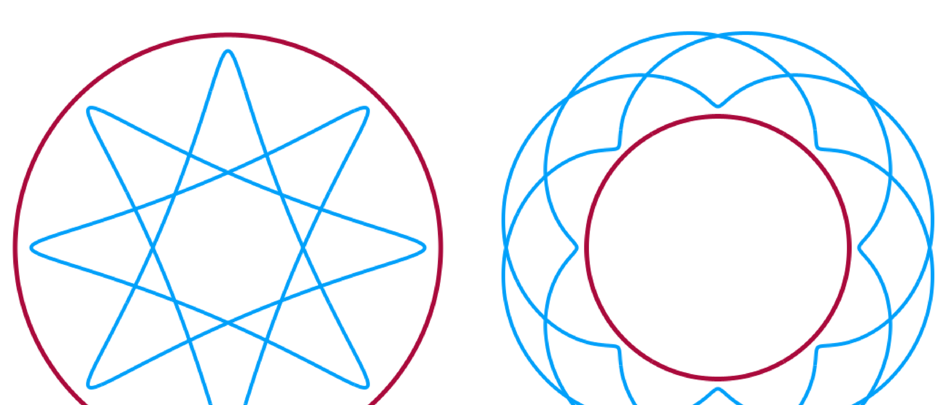


Figure: A hypotrochoid and an epitrochoid made with $R = 8$, $r = 3$, and $p = \frac{4}{5}$.

Tangles

A **tangle** is a smooth, simple, closed curve made up of circular arcs with constant radii and central angle. We can describe tangles using a series of ± 1s that designate the orientation of each tangle piece.

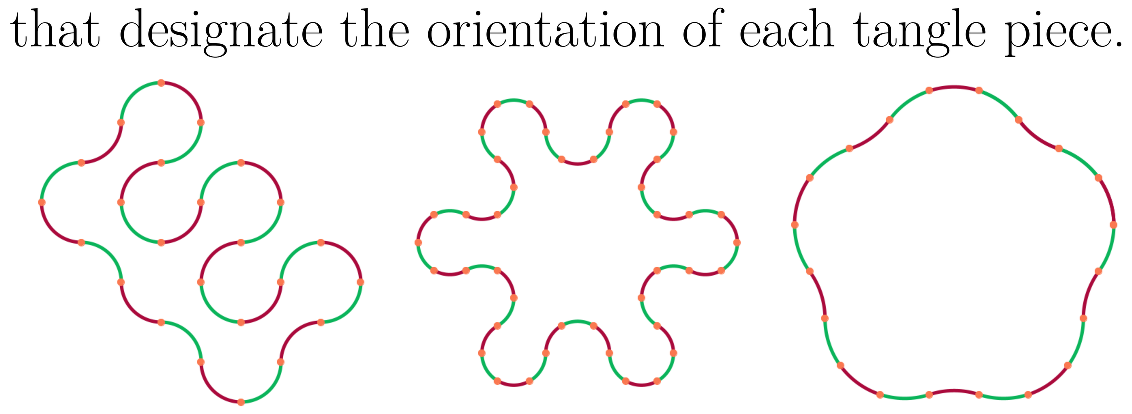


Figure: Some examples of tangles

Tangle Parameters

- $\vec{t} = \langle t_1, t_2, t_3, \ldots, t_m \rangle$; the sequence of ± 1s that define the tangle
- $\frac{\pi}{n}$; the central angle of each tangle piece, $n \geq 2$
- m; the number of tangle pieces

In this project we consider a generalization of hypotrochoids in which the wheel rolls around the interior of a tangle, calling these curves **tangloids**!

Tangloids

A tangloid is a piecewise defined curve in which every piece is a portion of a hypotrochoid or an epitrochoid.

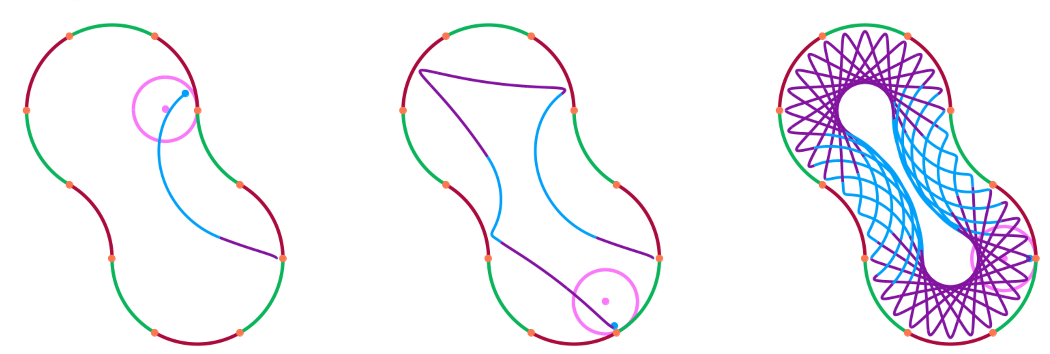


Figure: For this tangloid, hypotrochoid segments are purple and epitrochoid segments are blue.

Characteristic Ratio

The ratio of the length of the tangle to the circumference of the wheel is an important quantity we call the **characteristic ratio** of the tangloid.

$$\frac{mR}{2nr} = \frac{q}{v}$$

- The tangloid will be a closed curve if and only if $\frac{mR}{2nr}$ is rational.
- $\frac{q}{v}$ is the reduced form of $\frac{mR}{2nr}$.
- The wheel must make v revolutions around the full tangle in order to close.
- This ratio depends only on the number of tangle pieces, not on their configuration.

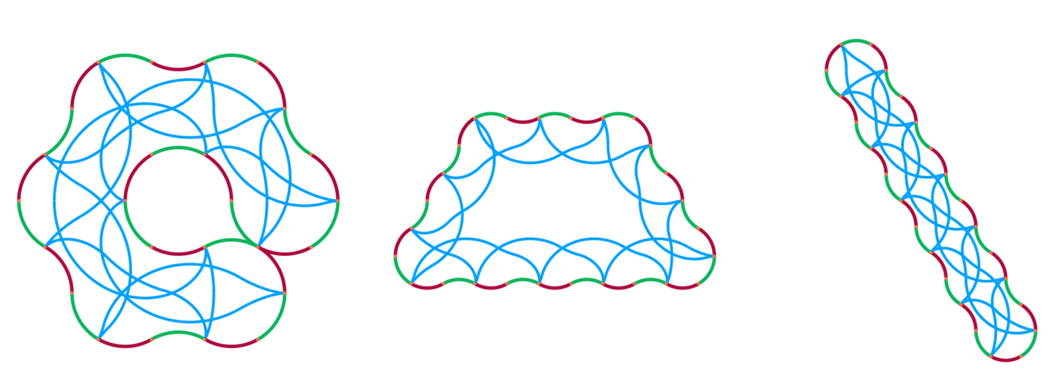
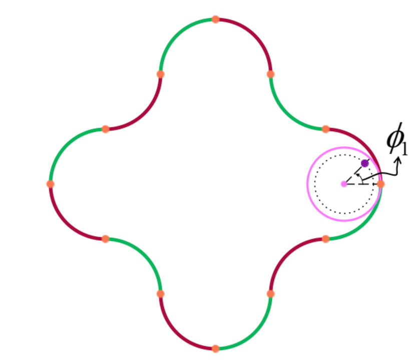


Figure: Three tangloids from different tangles with 26 pieces. Each has a characteristic ratio of $\frac{13}{2}$.

Tangloid Families

When we generate a tangloid, we put our pen at a specified distance from the center of the wheel. We then designate an angle ϕ_1 for the initial position of the pen.



The characteristic ratio is independent of the choice of ϕ_1. For a specific tangle, all tangloids that have the same characteristic ratio are part of the same **"tangloid family"**.

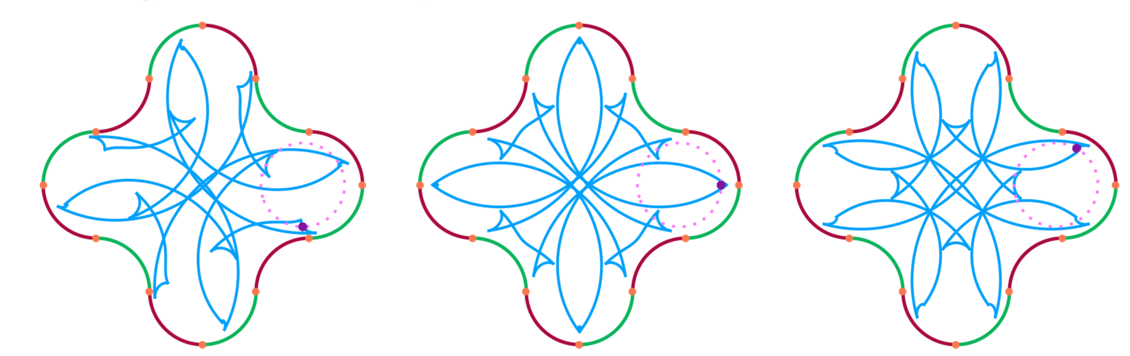


Figure: Three tangloids in one family, with various values for ϕ_1, all share a characteristic ratio of $\frac{8}{3}$.

Rotational Symmetry

A tangloid will inherit j-fold rotational symmetry of the tangle if the numerator of the characteristic ratio is divisible by j. As a consequence, *all* members of a tangloid family share the rotational symmetry of the tangle, or *none* of them do.

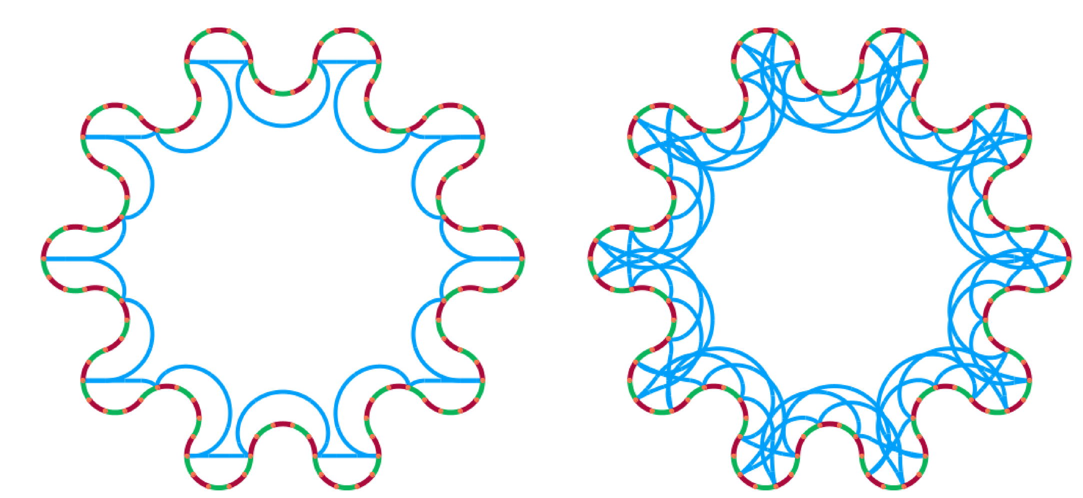


Figure: Tangloids with 2- and 5-fold symmetry constructed from a tangle with 10-fold symmetry.

Tangle and Tangloid Equations

$$T_k(\theta) = C_k + Re^{i(\alpha_k + t_k\theta)} \qquad 0 \leq \theta \leq \pi/n \text{ and } 1 \leq k \leq m$$

$$TG_k(\theta) = C_k + (R - t_k r)\, e^{i(\alpha_k + t_k\theta)} + pre^{i\left(t_k\theta - \frac{R}{r}\theta + \phi_k\right)} \qquad 0 \leq \theta \leq \pi/n \text{ and } 1 \leq k \leq mv$$

T_k: k^{th} tangle piece
TG_k: k^{th} tangloid piece
R: radius of each tangle piece
r: radius of the wheel
t_k: ± 1 that defines the direction for the k^{th} tangle piece
C_k: center of the k^{th} tangle piece
α_k: starting angle for the k^{th} tangle piece
p: portion of r where the pen lies
ϕ_k: starting angle for the pen at the kth tangle piece

Theorem

Suppose we have a tangle $\mathcal{T}$ made up of m circular arcs, each with a central angle $\frac{\pi}{n}$, radius R, with $\mathcal{T}$ having j-fold rotational symmetry. If we generate the tangloid family made from rolling a wheel with radius r around $\mathcal{T}$, it will inherit the j-fold symmetry from $\mathcal{T}$ if $j | q$ when $\frac{mR}{2nr} = \frac{q}{v}$.

Future Research

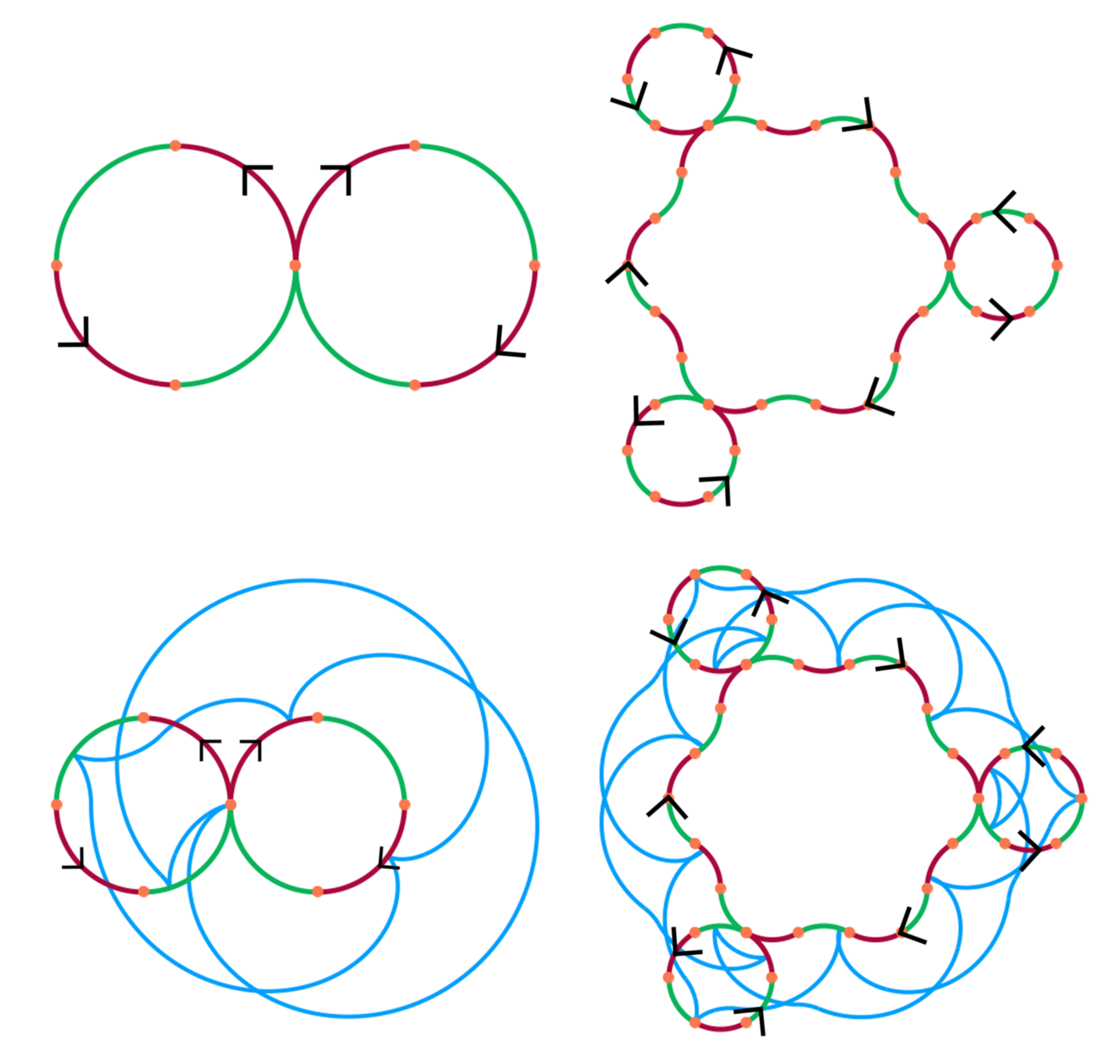


Figure: Two tangles with self intersections and their resulting tangloids. The rotational symmetry now must to take the direction of the curve into account.

References

[1] T.F. Banchoff and S.T. Lovett. *Differential Geometry of Curves and Surfaces.* A K Peters, 2010.

[2] Seth Colbert-Pollack, Micah Fisher, and Carol Schumacher. All tangled up. *Math Horizons*, 29(2):8–11, 2022.

[3] M. Lovrić. *Vector Calculus.* John Wiley & Sons, 2007.

Acknowledgements

I am beyond grateful for all of the guidance and support that Professor Schumacher provided. I would also like to thank the Kenyon Summer Science Scholars program for funding this work. This is an extension of research done by Seth Colbert-Pollack and Micah Fisher.